MECHANISM SYNTHESIS

I. Introduction

Mechanism synthesis is the process of generating the geometry of a mechanism that will perform a specific task. Here geometry generation means that the lengths of the individual links that make up the mechanism are determined. For this course the focus will be on planar four bar mechanisms. Figure 1 shows the generic four bar mechanism.

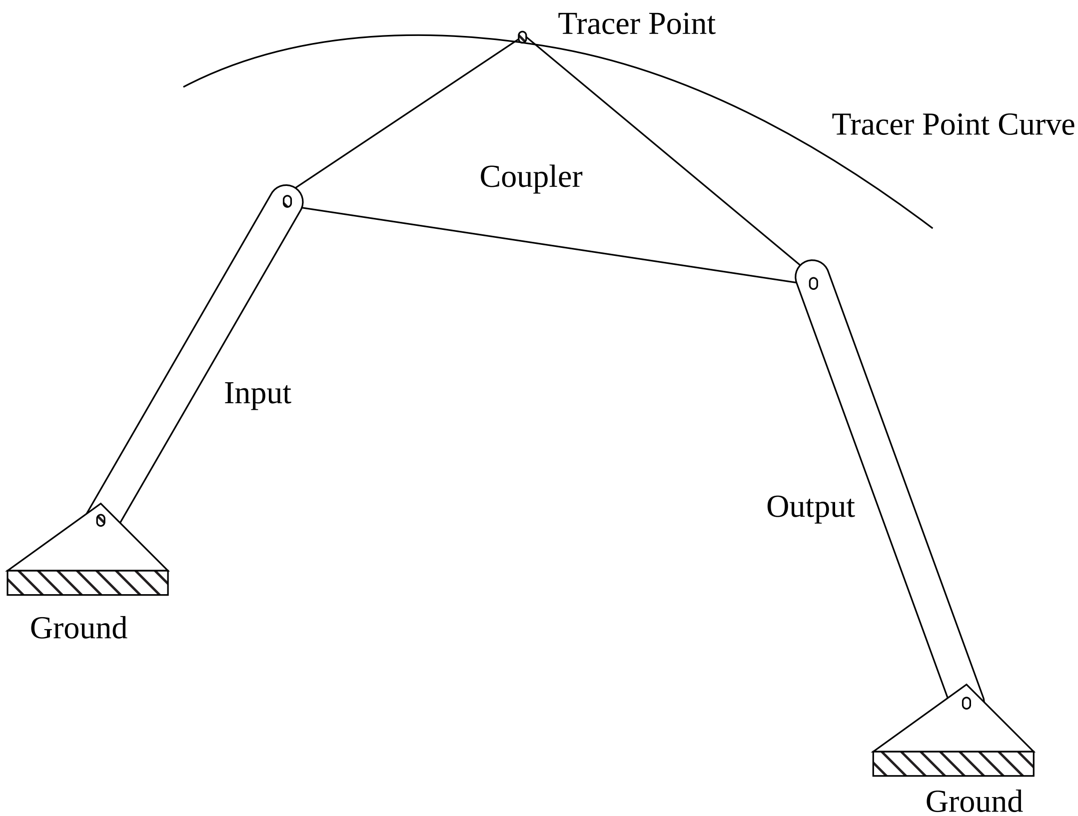


Figure 1 Generic Four Bar Mechanism

There are three broad categories that will be considered in the synthesis process:

1) Function Generation - defines the relationship between the position of the output link and the position of the input link. The motion that the coupler link goes through is not of concern. Examples of function generation include the automobile accelerator, the control stick in an aircraft, and the pistons in an engine.

2) Path Generation - defines locations through which the tracer point on the coupler will pass. The orientation of the coupler link is not important, but the time at which the tracer point passes the prescribed locations may be important. In this latter case the synthesis problem is path generation with prescribed timing.

3) Motion Generation - defines locations through which the tracer point passes and the orientation of the coupler link at those locations. Examples of motion generation include the power lift gate on a truck, the lift mechanism on a dumpster truck, and the windshield wipers on an automobile.

For all three synthesis types the prescribed conditions the mechanism must satisfy are called the precision positions. For function generation the precision positions consist of the pairs of input (β_i) and output (γ_i) angles the mechanism must meet. For path generation the precision positions are the pairs of coordinates (x_i, y_i) the tracer point must go through. In addition, for prescribed timing the angle of the input (β_i) will also be specified. In the case of motion generation the precision positions are the coordinates the tracer point passes through (x_i, y_i), as well as, the orientation of the coupler (α_i).

II. Vector Notation Review

Before the derivation of the equations for synthesizing a four bar mechanism are presented, a brief review of some vector notation is in order. Figure 2 shows two vectors and the relationship between them. For the present discussion the vectors are of equal length.

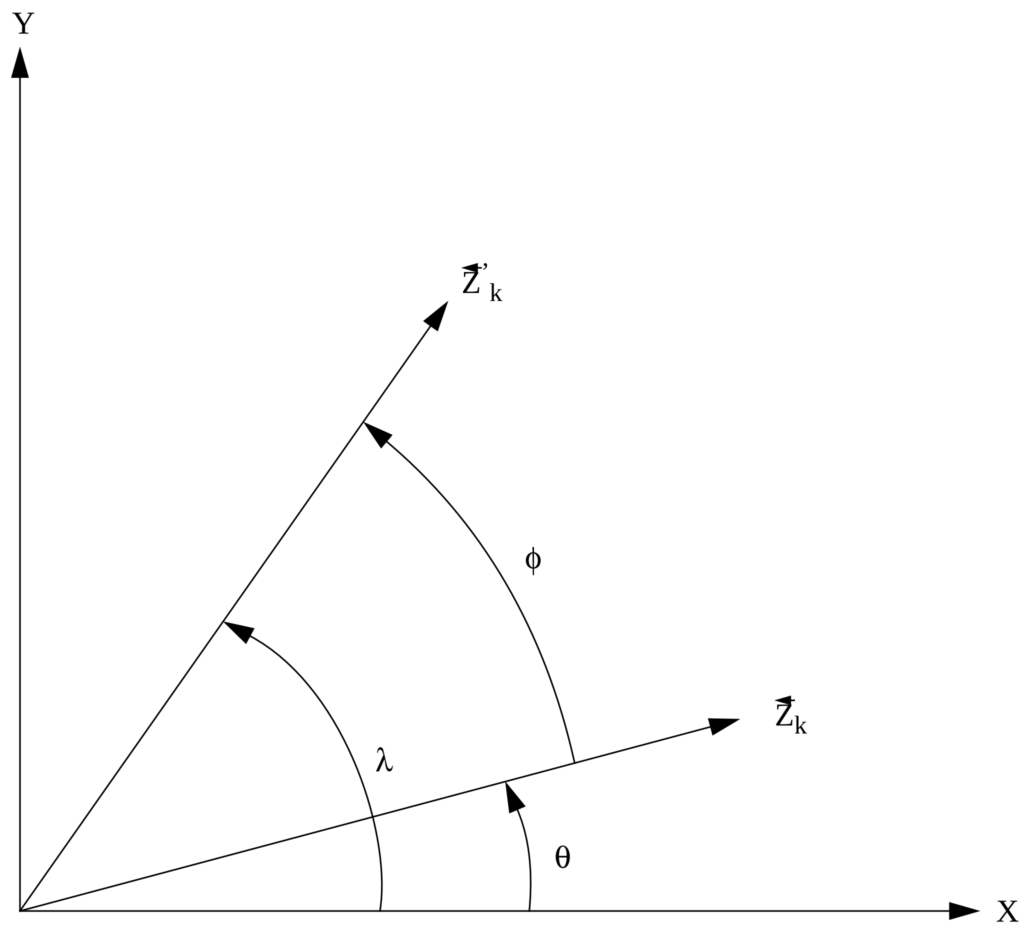


Figure 2 Vector Notation

The equations describing the vectors are:

$$\bar{Z}_k = Z_k(\cos\theta\hat{i} + \sin\theta\hat{j}) \tag{1}$$

and

$$\bar{Z}'_k = Z_k(\cos\lambda\hat{i} + \sin\lambda\hat{j}) \ . \tag{2}$$

where the vectors are expressed in terms of the unit vectors, $\hat{i}$ and $\hat{j}$. Another way to represent a vector is through the use of complex notation as shown below:

$$Z_k e^{i\theta} = Z_k(\cos\theta + i\sin\theta) \tag{3}$$

and

$$Z'_k e^{i\lambda} = Z_k(\cos\lambda + i\sin\lambda). \tag{4}$$

In this form, the vector is represented by a scalar whose real part represents the component that is directed along the *x* axis and whose imaginary part represents the magnitude of the component directed along the *y* axis. We also make use of the Euler identity,

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (5)$$

Now, the relationship between the angles is $\lambda = \theta + \phi$ which makes equation 4 become:

$$Z'_k e^{i\lambda} = Z_k e^{i(\theta+\phi)} = Z_k[\cos(\theta+\phi) + i\sin(\theta+\phi)] \ . \qquad (6)$$

Using the double angle relationships for *sin* and *cos*, equation 6 becomes:

$$Z'_k e^{i(\theta+\Phi)} = Z_k(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\,, \qquad (7)$$

which simplifies to

$$\bar{Z}'_k = \bar{Z}_k e^{i\phi} \ . \qquad (8)$$

The end result being the original vector rotated through the angle ϕ. Had the vector changed length in addition to rotating the new vector would be described by:

$$\bar{Z}'_k = \bar{Z}_k \rho e^{i\phi} \ , \qquad (9)$$

where

$$\rho = Z'_k / Z_k \ . \qquad (10)$$

For the new vector the stretch and the rotation would be independent.

III. Loop Equations

At this point a description of the equations and process for synthesizing a mechanism will be presented. Figure 3 gives a schematic representation of the four bar linkage from figure 1 with all of the vectors that represent the mechanism labeled. The four bar mechanism can be broken into two dyads, $\bar{Z}_2$ and $\bar{Z}_5$ which make up the input dyad, and $\bar{Z}_4$ and $\bar{Z}_6$ which make up the output dyad. Specifying these two dyads is enough to determine the entire mechanism because the remaining vectors $\bar{Z}_1$ and $\bar{Z}_3$ can be derived from:

$$\bar{Z}_3 = \bar{Z}_5 - \bar{Z}_6 \qquad (11)$$

and

$$\bar{Z}_1 = \bar{Z}_2 + \bar{Z}_3 - \bar{Z}_4 \qquad (12)$$

The angles between $\bar{Z}_3$, $\bar{Z}_5$, and $\bar{Z}_6$ are fixed, as these vectors make up the rigid coupler link. Figure 4 shows the four bar mechanism in two successive positions. The equations that are used to generate solutions for the input and output dyads are called loop equations. The loop begins at the ground pivot and proceeds out along the primed dyad vectors along the displacement vector from the second position to the first position and then back along the unprimed dyad vectors to the ground pivot.

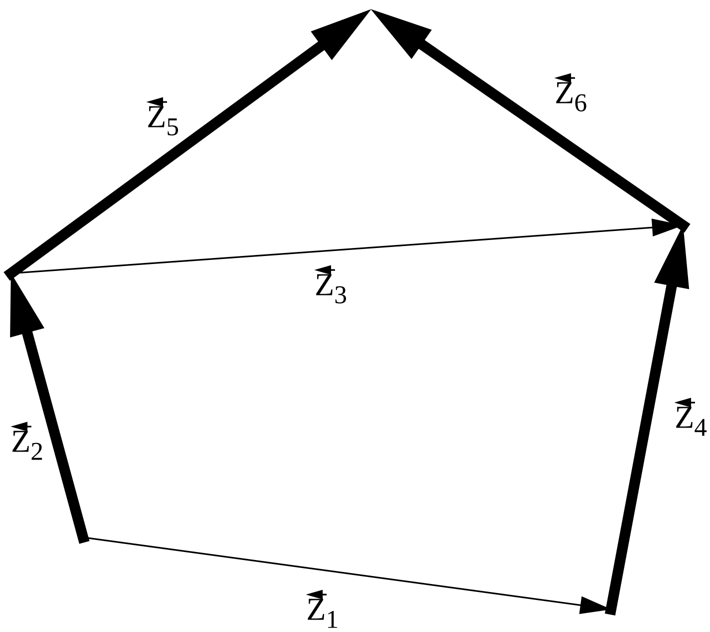


Figure 3 Four Bar Vectors

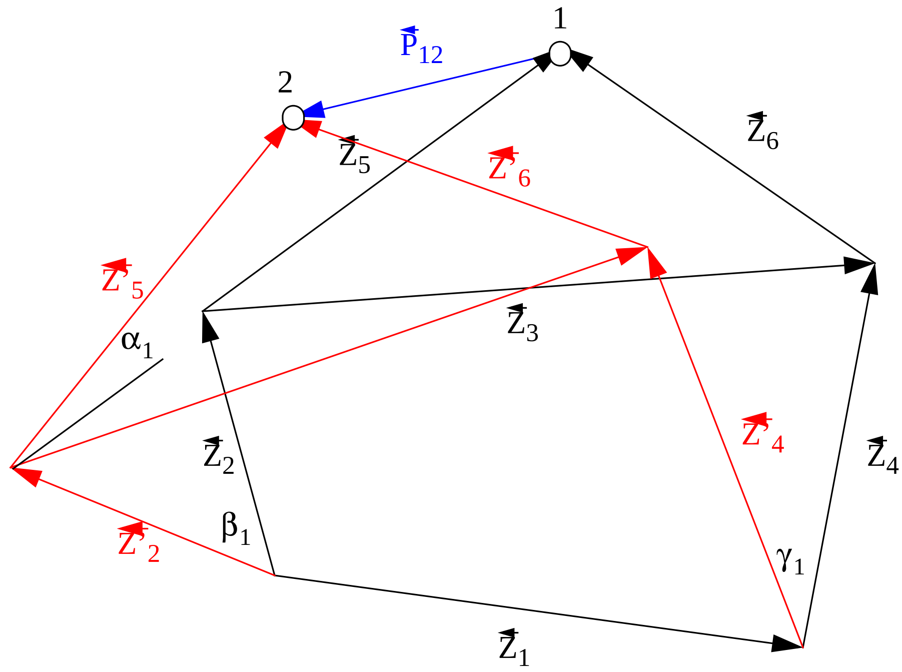


Figure 4 Vector Loops

For the input and output dyads the loop equations become:

$$\bar{Z}'_2 + \bar{Z}'_5 - \bar{P}_{12} - \bar{Z}_5 - \bar{Z}_2 = 0 \tag{13}$$

$$\bar{Z}'_4 + \bar{Z}'_6 - \bar{P}_{12} - \bar{Z}_6 - \bar{Z}_4 = 0 \ . \tag{14}$$

Moving the displacement vector to the right hand side and using the relationship between the primed and unprimed vectors from the previous review of vector notation the loop equations become:

$$\bar{Z}_2 e^{i\beta_1} + \bar{Z}_5 e^{i\alpha_1} - \bar{Z}_5 - \bar{Z}_2 = \bar{P}_{12} \tag{15}$$

$$\bar{Z}_4 e^{i\gamma_1} + \bar{Z}_6 e^{i\alpha_1} - \bar{Z}_4 - \bar{Z}_6 = \bar{P}_{12} \ . \tag{16}$$

These equations can be simplified to:

$$\bar{Z}_2(e^{i\beta_1} - 1) + \bar{Z}_5(e^{i\alpha_1} - 1) = \bar{P}_{12} \tag{17}$$

$$\bar{Z}_4(e^{i\gamma_1} - 1) + \bar{Z}_6(e^{i\alpha_1} - 1) = \bar{P}_{12} \ . \tag{18}$$

Equations 17 and 18 are commonly referred to as the standard form loop equations, and these are solved for the vectors $\bar{Z}_2$, $\bar{Z}_5$, $\bar{Z}_4$, and $\bar{Z}_6$. Each loop equation can be broken into two scalar equations. Table 1 lists what would typically be known and unknown for the different types of synthesis problems.

Table 1: Synthesis Problem Variables

Input Dyad		
Synthesis Type	Known	Unknown
Function	$\bar{Z}_6=0$, β_1, γ_1	$\bar{Z}_2$, $\bar{Z}_5$, α_1, $\bar{P}_{12}$
Path	$\bar{P}_{12}$	$\bar{Z}_2$, $\bar{Z}_5$, β_1, α_1
Motion	α_1, $\bar{P}_{12}$	$\bar{Z}_2$, $\bar{Z}_5$, β_1
Output Dyad		
Synthesis Type	Known	Unknown
Function	$\bar{Z}_6=0$, β_1, γ_1	$\bar{Z}_4$, $\bar{P}_{12}$
Path	$\bar{P}_{12}$	$\bar{Z}_4$, $\bar{Z}_6$, γ_1
Motion	α_1, $\bar{P}_{12}$	$\bar{Z}_4$, $\bar{Z}_6$, γ_1

In all cases there are more unknowns than there are equations. This gives rise to the situation where there can be multiple solutions to the problem. This multiplicity of solutions is obtained by making free choices for enough of the unknowns that the equations available can be solved for the remaining unknowns. This is design where the engineer has to generate solutions to the problem and then pick the best one from the solutions available. In the previous table only two precision positions where considered. Two precision positions results in one loop equation for each of the dyads, input and output. Each additional precision position can be used to generate another loop equation. Table 2 summarizes the relationship between precision positions and free choices for motion generation.

Table 2: Free Choices vs. Precision Positions for Motion Generation

Precision Positions	Displacement Vectors	Loop Equations	Scalar Equations	Unknowns	Free Choices
2	1	2	4	10	6
3	2	4	8	12	4
4	3	6	12	14	2
5	4	8	16	16	0

At this point the loop equations have been presented but the corresponding scalar equations have not been shown. The general form of the loop equation can be written as:

$$\bar{A}(e^{i\Psi} - 1) + \bar{B}(e^{i\Omega} - 1) = \bar{C} , \tag{19}$$

$$(A_x + iA_y)[(\cos\Psi - 1) + i\sin\Psi] + (B_x + iB_y)[(\cos\Omega - 1) + i\sin\Omega] = C_x + iC_y . \tag{20}$$

This gives rise to two scalar equations of the form:

$$A_x(\cos\Psi - 1) - A_y\sin\Psi + B_x(\cos\Omega - 1) - B_y\sin\Omega = C_x , \tag{21}$$

$$A_x\sin\Psi + A_y(\cos\Psi - 1) + B_x\sin\Omega + B_y(\cos\Omega - 1) = C_y . \tag{22}$$

Some of the implications of the solution of the synthesis problem are:

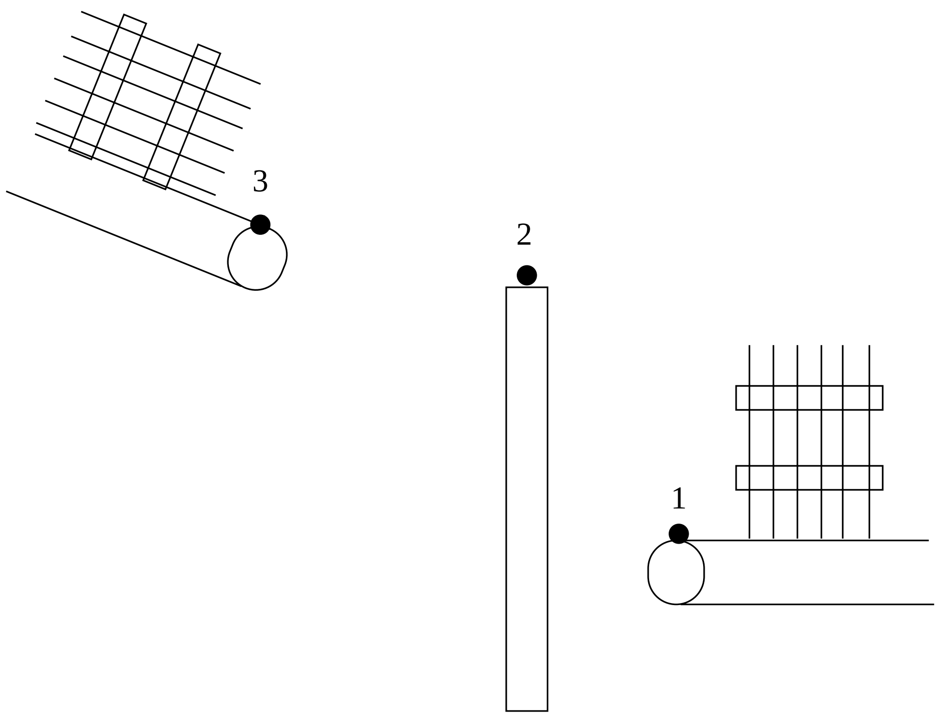
- For a given solution the mechanism can be assembled at all of the precision positions, however there is no guarantee that the mechanism can move from one position to the next.

- A particular solution may be able to move between all of the precision positions, but it may not be suitable because of size or transmission angle considerations.

- Using the free choices to select angles will linearize the problem which makes obtaining a solution easier.

IV. Example: 3 Point Motion Generation

The J. E. Smith Box Co. is currently refurbishing its plant in Millersville, MD. The plant manager comes to you (the plant engineer) and asks for an automated arrangement for moving the product from the wrapping line to the stamping line. You go down to the shop floor and see that the product he is referring to is bundles of flattened cartons that are banded together and then stamped with the proper inventory information. You also observe the layout of the wrapping and stamping lines, which are shown below:



In addition to the locations of the lines, you observe that the bundles on the wrapping line are perpendicular to the conveyor and on the stamping line they are parallel. In addition, the bundles must clear a partition between the two lines. Based on these observations the problem formulation is to design a mechanism that will transfer the bundles from location 1 to location 3, while clearing location 2 and rotating the bundles through an angle α. The details for the problem are:

position 1 (0,0)
position 2 (-6,11)
position 3 (-17,13)
rotation α 68^o, choose an intermediate angle of 22^o.

The known quantities become: $\bar{P}_{12} = -6 + i11\ ,\ \alpha_1 = 22^o$

$$\bar{P}_{13} = -17 + i13\ ,\ \alpha_2 = 68^o$$

The unknown quantities are $\overline{Z}_2$, $\overline{Z}_5$, $\overline{Z}_4$, $\overline{Z}_6$, β_1, β_2, γ_1, and γ_2. Based on equation 20, the loop equations become:

Input Dyad

$$(Z_{2x} + iZ_{2y})[(\cos\beta_1 - 1) + i\sin\beta_1] + (Z_{5x} + iZ_{5y})[(\cos(22^o) - 1) + i\sin(22^o)] = -6 + i11$$

$$(Z_{2x} + iZ_{2y})[(\cos\beta_2 - 1) + i\sin\beta_2] + (Z_{5x} + iZ_{5y})[(\cos(68^o) - 1) + i\sin(68^o)] = -17 + i13$$

Output Dyad

$$(Z_{4x} + iZ_{4y})[(\cos\gamma_1 - 1) + i\sin\gamma_1] + (Z_{6x} + iZ_{6y})[(\cos(22^o) - 1) + i\sin(22^o)] = -6 + i11$$

$$(Z_{4x} + iZ_{4y})[(\cos\gamma_2 - 1) + i\sin\gamma_2] + (Z_{6x} + iZ_{6y})[(\cos(68^o) - 1) + i\sin(68^o)] = -17 + i13$$

Using equations 21 and 22, these loop equations give rise to the following scalar equations:

Input Dyad

$$Z_{2x}(\cos\beta_1 - 1) - Z_{2y}\sin\beta_1 + Z_{5x}(\cos(22^o) - 1) - Z_{5y}\sin(22^o) = -6$$

$$Z_{2x}\sin\beta_1 + Z_{2y}(\cos\beta_1 - 1) + Z_{5x}\sin(22^o) + Z_{5y}(\cos(22^o) - 1) = 11$$

$$Z_{2x}(\cos\beta_2 - 1) - Z_{2y}\sin\beta_2 + Z_{5x}(\cos(68^o) - 1) - Z_{5y}\sin(68^o) = -17$$

$$Z_{2x}\sin\beta_2 + Z_{2y}(\cos\beta_2 - 1) + Z_{5x}\sin(68^o) + Z_{5y}(\cos(68^o) - 1) = 13$$

Output Dyad

$$Z_{4x}(\cos\gamma_1 - 1) - Z_{4y}\sin\gamma_1 + Z_{6x}(\cos(22^o) - 1) - Z_{6y}\sin(22^o) = -6$$

$$Z_{4x}\sin\gamma_1 + Z_{4y}(\cos\gamma_1 - 1) + Z_{6x}\sin(22^o) + Z_{6y}(\cos(22^o) - 1) = 11$$

$$Z_{4x}(\cos\gamma_2 - 1) - Z_{4y}\sin\gamma_2 + Z_{6x}(\cos(68^o) - 1) - Z_{6y}\sin(68^o) = -17$$

$$Z_{4x}\sin\gamma_2 + Z_{4y}(\cos\gamma_2 - 1) + Z_{6x}\sin(68^o) + Z_{6y}(\cos(68^o) - 1) = 13$$

There are eight equations available to solve for the twelve unknowns. Therefore, free choices will be used to select values for four of the variables. Recall that selecting values for the unknown angles will make the system of equations linear. The following values are chosen for the β's, and the γ's:

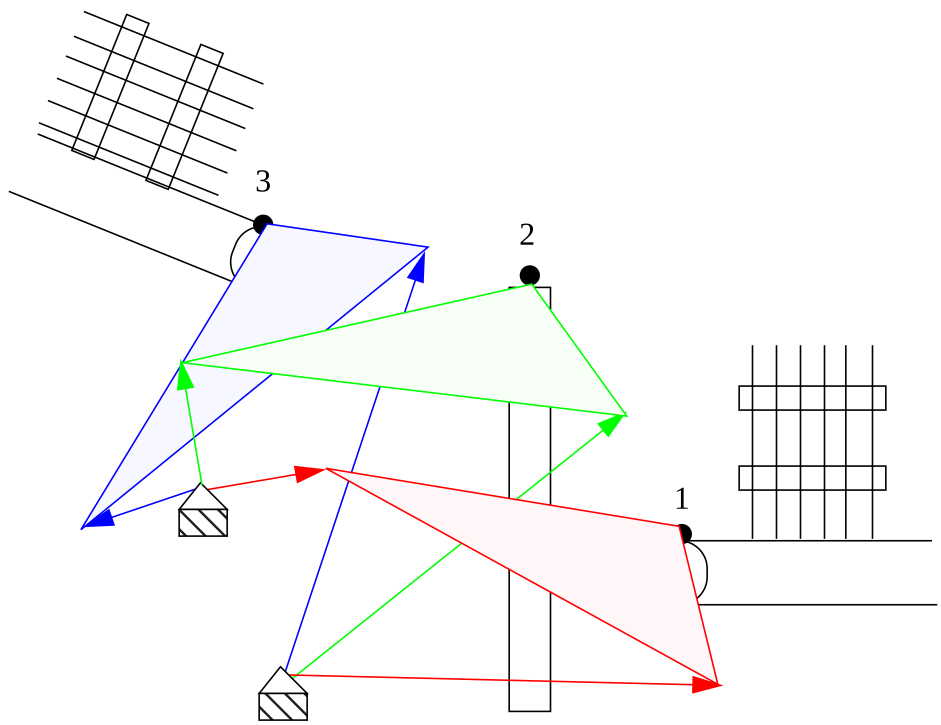
$$\beta_1 = 90^o, \quad \gamma_1 = 40^o$$

$$\beta_2 = 198^o , \quad \gamma_2 = 73^o .$$

the solution to this system of equations gives the following values for the vectors that make up the input and output dyads:

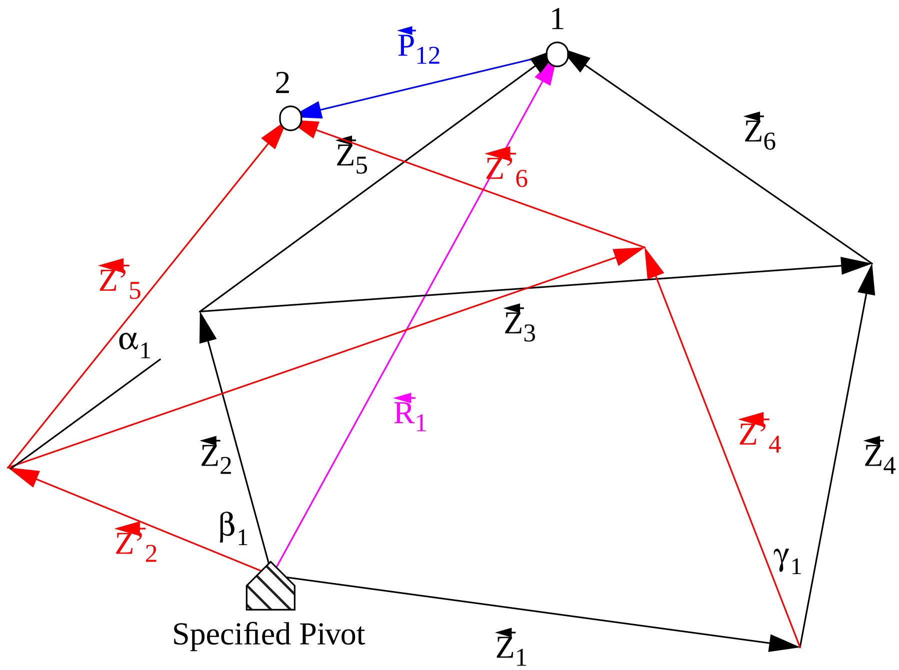
$$Z_{2x} = 5.755 , \quad Z_{2y} = 0.481$$
$$Z_{5x} = 14.611 , \quad Z_{5y} = -3.470$$
$$Z_{4x} = 18.375 , \quad Z_{4y} = -0.661$$
$$Z_{6x} = -1.421 , \quad Z_{6y} = 5.952$$

The mechanism is located in space by starting at the first precision position and drawing lines back along the solutions given for the vectors that make up the input and output dyads. The resulting mechanism is shown in all three positions in the figure below.



V. Specified Pivot Location

In some instances the locations of the ground pivots must be confined to specific locations. When this is the case the specification of the pivot location adds an additional loop equation for the dyad and takes two free choices for that dyad. The new loop runs from the specified pivot location out the along the dyad to the first precision position and then back to the specified pivot location along the vector $\overline{R}$. This vector is determined by taking the difference between the coordinates of the first precision position and the specified pivot location. This is illustrated in the figure below.



The additional loop equation is:

$$\bar{Z}_2 + \bar{Z}_5 = \bar{R}_1 \ .$$

This loop equation provides two additional scalar equations but does not introduce any new unknowns. For this reason the number of free choices for the dyad is reduced by two. So, in the previous example if the ground pivot locations had been specified there would have been no free choices available. There would have been six scalar equations for each dyad which would have produced twelve equations for the twelve unknowns. As a result the system of equations would have been nonlinear, because it would not have been possible to eliminate the angles.